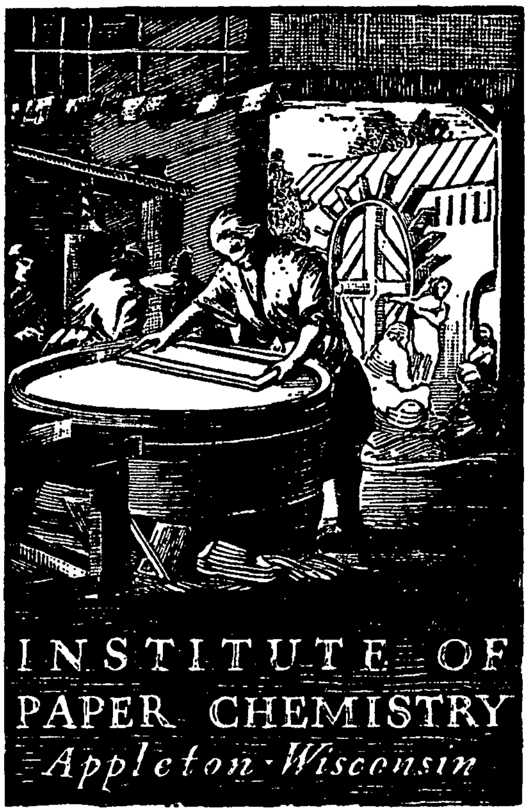

RELATIONSHIP BETWEEN SACK PERFORMANCE AND THE PROPERTIES OF SACK PAPER

PART II. A Theory for the Behavior of Regular Sack Paper in Repeated Uniaxial Tension

Project 2033

Report Eighteen

A Progress Report

to

MULTIWALL SHIPPING SACK PAPER MANUFACTURERS

July 25, 1961

THE INSTITUTE OF PAPER CHEMISTRY

Appleton, Wisconsin

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PAPER IN REPEATED UNIAXIAL TENSION

SUMMARY

1. The laboratory repeated impact performance of multiwall sacks appears to be dependent on the fatigue and energy absorption characteristics of the sack paper from which they are fabricated (in addition to design, fabrication efficiency and commodity characteristics).

2. The <u>fatigue</u> <u>life</u> of a structure (e.g., a sack) or of a material (e.g., sack paper) is conventionally defined as the number of applications of stress or strain which may be sustained before failure.

3. It is reasonable to expect that the fatigue life of a sack (that is, its repeated impact performance) may be directly related to the fatigue life of the sack paper in tension, the latter evaluated under appropriate conditions.

4. A mathematical theory is presented for the uniaxial tensile fatigue life of regular sack paper in terms of its basic load-elongation properties. The fatigue life equation for the case of constant applied elongation (which is apparently equivalent to constant applied energy) is

$$N = \frac{\log \left\{ \dfrac{(e_a/e_v) - \alpha - (\beta - \alpha)(e_o/e_v)}{(1 - \alpha)(e_a/e_v) - (\beta - \alpha)(e_o/e_v)} \right\}}{\log (1 - \alpha)}$$

where $\underline{N}$ = fatigue life

$\underline{e_a}$ = applied elongation

$\underline{e_v}$ = virgin elongation

$\underline{e}_o$ = proportional limit elongation

α = $\underline{S}_p/\underline{S}_r$

β = $\underline{S}_o/\underline{S}_r$

$\underline{S}_o$ = initial slope of virgin load-elongation curve

$\underline{S}_p$ = plastic slope of virgin load-elongation curve

$\underline{S}_r$ = slope of average reload curve in repeated tension

It may be seen that the theoretical uniaxial fatigue life of regular sack paper is dependent on the applied elongation and on five load-elongation parameters (four of which are defined in terms of straight-line approximations to the load-elongation curve). Suggestions are given for the experimental determination of the slope of the reload curves for purposes of practical application of the fatigue life equation.

5. The fatigue life equation offers the prospect that the uniaxial fatigue performance of sack paper may be predicted from data obtained from the conventional tension test rather than from fatigue type tests. Moreover, the effect of fiber and sheet characteristics on tension load-elongation properties may be translated thereby to potential fatigue performance.

6. It is shown that curves of theoretical fatigue life <u>versus</u> applied elongation based on the preceding equation resemble the conventional $\underline{S}$-$\underline{N}$ diagrams for other materials, except that the magnitude of fatigue life of interest to sack paper is several orders of magnitude lower than for most other materials.

7. The fatigue life theory embraces the special case of the virgin tensile test.

8. Endurance limit is conventionally defined as the limiting value of stress or strain below which a material can presumably sustain an infinite number of stress cycles. In broad terms, the higher the endurance limit, the better is the fatigue performance of a material.

9. A relatively uncomplicated equation is derived for the endurance limit, e_∞, of sack paper in uniaxial tensile fatigue, namely,

$$e_\infty = \alpha \, e_v + (\beta - \alpha) \, e_o$$

The endurance limit possibly has utility as an index of the fatigue performance of sack paper.

10. It is shown that the endurance limit, e_∞, is merely the elastic component of the virgin elongation and, in general, is greater than the proportional limit elongation.

11. The theory indicates that the endurance limit of a viscoelastic material ($\alpha > 0$) is always higher than that of an ideally plastic material ($\alpha = 0$), all other pertinent factors being equal. Thus, it is theoretically better to have a steep rather than shallow plastic slope in the virgin load-elongation curve of a sack paper, all other factors being equal.

12. Sack paper fails in uniaxial fatigue because of the progressive deterioration of the virgin stretch (and tensile energy absorption), as discussed in the preceding report of this series. Thus, the deterioration characteristics of sack paper are of importance to an understanding of fatigue life.

13. An equation is derived for the relationship between residual elongation and the number of strain applications to which the sack paper has been subjected. Although experimental verification of the residual elongation formula

(as well as the fatigue life equation) will be treated in a forthcoming report, it is already apparent that the formula is compatible in its essentials with the earlier experimental studies of deterioration. This observation lends credence to the present theoretical concept of fatigue behavior.

14. Work is now in progress for the purpose of confirming this theoretical study.

15. Recommendations are given for future work relative to (a) extension, simplification, and possible improvement of the fatigue theory, and (b) application of sack paper fatigue considerations to sack impact performance.

INTRODUCTION

The relationship between the performance of multiwall sacks and the properties of the sack paper has challenged the manufacturers of multiwall sacks and sack papers for many years. The only general agreement appears to be that the conventional paper tests, such as bursting strength, tensile strength, stretch, tear, etc., do not adequately reveal the properties of the paper which govern sack performance in the laboratory sack impact test or in the field.

From the studies of sack impact which have been carried out in this laboratory and elsewhere in recent years, it seems quite apparent that sack performance is associated with the fatigue behavior of the sack paper (except in the special case where one impact is sufficient to rupture the bag). Moreover, consideration of the mechanics of the sack during impact suggests that tensile energy absorption (under fatigue conditions) is probably the pertinent material property on which to focus attention. That fatigue considerations should play an important role in laboratory sack impact behavior is almost self-evident, inasmuch as the failure of a material after repeated applications of a stress or strain is, by definition, a fatigue phenomenon. Therefore, knowledge of the fatigue behavior of sack paper would appear to be essential to an understanding of sack performance.

The views expressed above and the experimental evidence on which they are based are developed more fully in Reference (1). Experimental data acquired for regular 50-lb. kraft sack papers from a number of different manufacturers indicated that the laboratory impact performance of three-ply cement sacks was most highly correlated with paper tests which were of (a) a fatigue nature (Frag, Thwing-Albert impact fatigue) and (b) an energy type of test, or tests closely related to energy absorption (tensile energy absorption, stretch, Van der Korput dynamic energy, impulse).

These considerations and observations prompted a series of studies relative to the behavior of sack paper in repeated tension and the mechanisms leading to fatigue rupture of the paper. In an initial phase of the work, sack paper specimens taken from filled sacks which had been impacted various numbers of times exhibited a deterioration in their tensile properties (2). Shortly afterward Ihrman and Andersson (3) reported similar results. Pursuant to the interest expressed by the Technical Committee, the study of repeated tension behavior of sack paper was intensified in this laboratory. The next phase of the investigation was comprised of (a) a search of the moderately extensive literature on the rheological (viscoelastic) behavior of paper, and (b) an exploratory experimental study of the behavior of sack paper in repeated uniaxial tension under the controlled conditions of the laboratory tensile test (4). Finding that sack paper behavior was not at variance with the dominant rheological characteristics of paper in general, as reported in the literature, an effort was made in Reference (4) to construct a working hypothesis of the relationship between the fatigue life of sack paper in uniaxial tension and the load-elongation properties upon which fatigue life is dependent.

Fatigue life is defined (5,6) as the number of applications of stress (or strain) that a material can sustain before rupture under a given test condition. It may be appreciated that the laboratory sack impact test is a test of the fatigue life of a paper structure, just as the Frag and Thwing-Albert impact fatigue tests are essentially tests of the fatigue life of the sack paper. Although the results of the sack impact test are commonly expressed in units of accumulated safe inches of drop, they could be expressed equally well in terms of the number of drops prior to rupture, that is, fatigue life. In the case of the constant height drop test, the two measures of performance are directly proportional. For the progressive height drop test, there is a determinable relationship between the two measures of performance, namely,

$$N = \frac{-(2H - h) + \sqrt{(2H - h)^2 + 8hS}}{2h}$$

where $\underline{N}$ = fatigue life of sack, dimensionless.

$\underline{S}$ = accumulated safe inches of drop.

$\underline{H}$ = initial drop height, in.

$\underline{h}$ = difference in height between successive drops, in.

It is reasonable on physical grounds, therefore, that the fatigue life of a sack (i.e., the laboratory impact test) should be capable of being expressed in terms of the fatigue life of the sack paper under given conditions (in conjunction with other factors descriptive of the sack construction, the commodity and the impact conditions).

This present report is an attempt to give quantitative expression to the fatigue life hypothesis in accordance with a research objective approved by the Technical Committee (7), namely,

"C.2. Development of relationship (mathematical) between fatigue life of sack paper and the fundamental mechanical properties of the sack paper based on

a. Energy absorption concept.

b. Strain concept."

In view of the conceptual and experimental evidence (8) of the existance of biaxial stresses and strains in impacted multiwall sacks, it may be asked: Why the preoccupation with uniaxial tension? The answer to this question is twofold. First, a theory for uniaxial tensile fatigue behavior may serve as a stepping-stone to the more difficult problem of biaxial fatigue behavior. Stated differently, it would be presumptuous to expect the development of an adequate theory for biaxial fatigue if it were not possible to develop a theory for uniaxial fatigue. Secondly, the classical treatment of biaxial strength has been to relate it to the uniaxial strengths, in view of the advantage of the latter type of test with respect to both availability of equipment and accuracy. Assuming that a similar correspondence may be developed between biaxial and uniaxial fatigue behavior, a workable theory for the latter should be capable of serving the interests of sack paper technology.

It should be emphasized at the outset that the theoretical analysis of fatigue life presented in this report differs somewhat in its underlying philosophy from that which has been current in the rheological literature for paper, of which Reference (4) contains a representative bibliography. The latter approaches have largely followed the classical (and indeed sophisticated) concepts of rheology, frequently involving the formulation of mechanical analogies (springs, dashpots, etc.) to explain the response of a paper web to repeated tension, creep and relaxation. The approach followed in this report, on the other hand, is a somewhat more phenomenological approach, wherein an attempt is made to relate fatigue life (and tension deterioration) to the macroscopic load-elongation properties of sack paper. The properties concerned may be evaluated with conventional modern tensile testing equipment in the laboratory of the mill or converter.

It may be assumed that the aforementioned classical rheological approaches are directly ultimately to the role of fiber properties and sheet structure in the stress-strain behavior of paper. The worth of such studies and their possible application to paper technology is not to be denied. But as of the present time, the classical approach to the rheology of paper is fragmentary and not sufficiently developed to be of practical aid to the papermaker. It is believed, therefore, that there is a practical advantage to be gained from an approach to fatigue theory such as is embodied in this report in that it has as its objective the relationship between fatigue life and those currently measurable properties of sack paper such as tensile strength, stretch, and energy absorption.

The structure of this report is as follows: After a brief review of the fundamental concepts of repeated tension behavior [discussed in greater detail in Reference (4)], a mathematical expression is derived for the relationship between uniaxial fatigue life and the basic load-elongation properties of the sack paper. The theory is developed for the special case of repeated applications of a constant level of elongation. Inasmuch as previous work (4) indicated that a constant elongation process was equivalent to a constant work process for regular sack paper, there is no conflict with the tenet that the constant height drop test is a constant energy process.

Secondly, an equation is developed for the deterioration in stretch as a function of the number of strain applications to which the sack paper has been subjected. Deterioration in tensile properties may be regarded as the mechanism whereby repeated tension leads to fatigue failure. As such, deterioration characteristics are of considerable importance to an understanding of fatigue life of paper. It may be recalled that the experimental program of Reference (4) was oriented to a study of tension deterioration. A subsequent report will deal with the experimental verification of the theory.

BEHAVIOR OF SACK PAPER IN REPEATED TENSION

A load-elongation curve such as is obtained from a repeated uniaxial tension test (constant applied elongation) is sketched in Fig. 1. The first application of strain, $\underline{e}_a$, brought the specimen to the load-elongation state denoted by point $\underline{A}$, following the path $\overline{OPA}$. Thereafter, the specimen was unloaded at the same rate along the dashed-line path $\overline{AB'B}$, which includes a brief recovery period $\overline{B'B}$. The specimen was again strained in the amount $\underline{e}_a$ which brought the load-elongation state of the paper to a point denoted by point $\underline{D}$.

It may be noted from Fig. 1 that upon reloading (solid line curve) the curve passes quite near, but not precisely through, the previous unloading point $\underline{A}$. Various investigators have found slightly differing effects in this region of the curve, but in general the deviation is small. Most important, in the vicinity of this point the reload curve exhibits a marked change in slope. From points $\underline{A}'$ to $\underline{D}$ the curve is essentially that which the virgin specimen would have followed if the initial loading had not been interrupted at point $\underline{A}$ (9). The characteristic wherein the reload curve passes nearly through the previous unloading point $\underline{A}$ and then continues along the virgin branch of the tensile load-elongation curve may be described as a "memory" exhibited by the paper; that is, the paper seemingly remembers the terminal point of the preceding load curve and rejoins the virgin branch of the curve at that point. This characteristic is also displayed by many metallic materials; the cycling process is frequently referred to as "strain hardening" inasmuch as the material behaves essentially elastically to the point $\underline{A}$ (or $\underline{A}'$) during the second loading and in effect the limit of proportionality $\underline{P}$ is elevated to $\underline{A}$ (10).

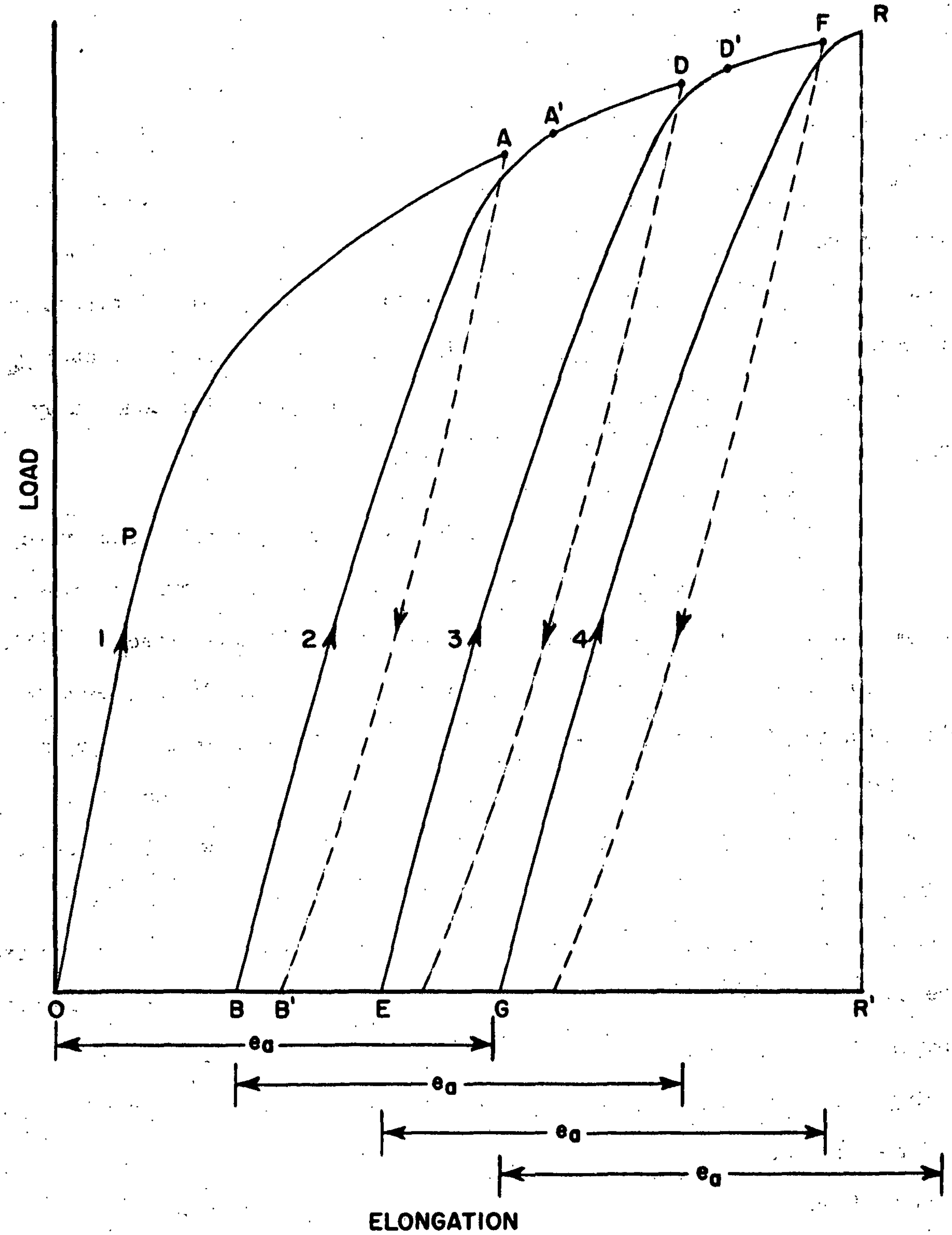


Figure 1. Typical Load-Elongation Curve of Sack Paper in Repeated Uniaxial Tension (Constant Applied Elongation)

When the loading was reversed at point D of Fig. 1, the load-elongation state of the paper followed a dashed-line path $\overline{DE}$. Subsequent reloading to a strain e_a brought the paper to a point F, via the path $\overline{ED'F}$, where $\overline{D'F}$ is again a portion of the virgin tensile curve.

When the cycling process is continued in the aforementioned manner, the specimen eventually ruptures when the point R is reached (four applications of elongation, e_a, in the case of Fig. 1). Experiments have shown that the rupture load is appreciably the same as the virgin tensile strength (4, 9, 11, 12, 13). Moreover, the total stretch $\overline{OR'}$ is very nearly equal to the virgin stretch (13, 14, 15, 16), provided the cycling is not interrupted by lengthy periods of relaxation (17). To a first approximation, therefore, the rupture point R is an invariant, that is, the paper ruptures at a load and elongation which are (a) independent of the stress-strain history of the cycling process and (b) equal to the virgin tensile and stretch (for a given rate of loading).

The invariance of the rupture point can be expected to hold only if the rate of strain application is maintained constant throughout the repeated tension test, as discussed at some length in Reference (4). Consequently, it is assumed in this case, and in the remainder of this report, that the rate of loading and unloading and the duration of the recovery period are all maintained at constant levels.

It may be noted from Fig. 1 that at the end of the first strain cycle, $\overline{OPAB}$, the residual stretch of the paper is $\overline{BR'}$; that is, the virgin stretch has been reduced by the amount $\overline{OB}$, which is termed the nonrecoverable elongation due to the first strain cycle. The second strain cycle, $\overline{BDE}$, results in a further increment of nonrecoverable elongation, $\overline{BE}$, so that the total non-recoverable stretch at the end of the second cycle is $\overline{OB} + \overline{BE} = \overline{OE}$, whereupon the available stretch in the paper is $\overline{ER'}$. It may be seen that each subsequent cycle adds an increment of nonrecoverable elongation, and reduces the available stretch of the paper. Eventually, the residual stretch is reduced sufficiently so that the subsequent application of strain, $\underline{e}_a$, exceeds the residual stretch and the specimen ruptures.

Similarly, the tensile energy absorption (work) of the virgin paper, proportional to area $\overline{OPRR'}$ of Fig. 1, is reduced by an amount proportional to area $\overline{OPA'B}$ as a result of the first cycle; that is, the residual tensile work after the first cycle is area $\overline{BA'RR'}$. The second cycle results in an additional increment of nonrecoverable work, area $\overline{BA'DE}$, so that after two cycles, the total nonrecoverable work is area $\overline{OPD'E}$ and the residual work is area $\overline{ED'RR'}$. On the fourth application of strain, $\underline{e}_a$, the associated applied work exceeds the residual work of the paper and rupture occurs. Thus, the final application of strain causes both the residual stretch and work to be exceeded, that is, the specimen ruptures. For that matter, the tensile strength is also exceeded on the final application of strain. Unlike stretch and work, however, there is no significant deterioration in tensile strength as a result of cycling.

Although the behavior described above was in terms of repeated application of a constant magnitude of strain, $\underline{e}_a$, the same concepts may be expected to hold for any other arbitrary mode of strain cycling, for example, progressively increasing or progressively decreasing applied strain. There are, of course, an infinite number of conceivable repeated strain processes, but the behavior of the paper may be expected to differ in degree rather than in principle for the various strain processes.

Moreover, the repeated tension process may be described in terms of applied work rather than applied strain, e.g., constant applied work or progressively increasing or decreasing applied work. The load-elongation behavior for an applied work process differs from that described above only in that the terminal point of each application of load (e.g., A, D, F) is determined by the area beneath the loading curve ($\overline{OPA}$, $\overline{BA'D}$, $\overline{ED'F}$). It should be mentioned that the investigation of Reference (4) revealed that for regular 50-lb. sack paper, a constant applied work process was equivalent to a constant applied strain process and conversely, which diminishes somewhat the number of distinctive repeated tension processes. Unpublished data from a limited study indicates that the aforementioned equivalence may not hold for extensible 50-lb. sack paper.

The points A and A′ of Fig. 1 are the points where the cycling history of the paper departs from and rejoins, respectively, the virgin branch of the tensile-elongation curve. The disparity between points A and A′ (and similarly between D and D′, etc.) almost always is observed in repeated tension testing. For the purpose of formulating a mathematical description of repeated tension behavior, however, it is convenient to idealize the tensile-elongation curve so as to exclude the disparity between points A and A′. This may be done by constructing a line through A which is tangent to the reload curve $\overline{BA'}$, as illustrated in Fig. 2. It may be seen that the error incurred by this idealization is generally in the nature of a modest overestimation of the tensile work applied during the strain application $\overline{BAD}$.

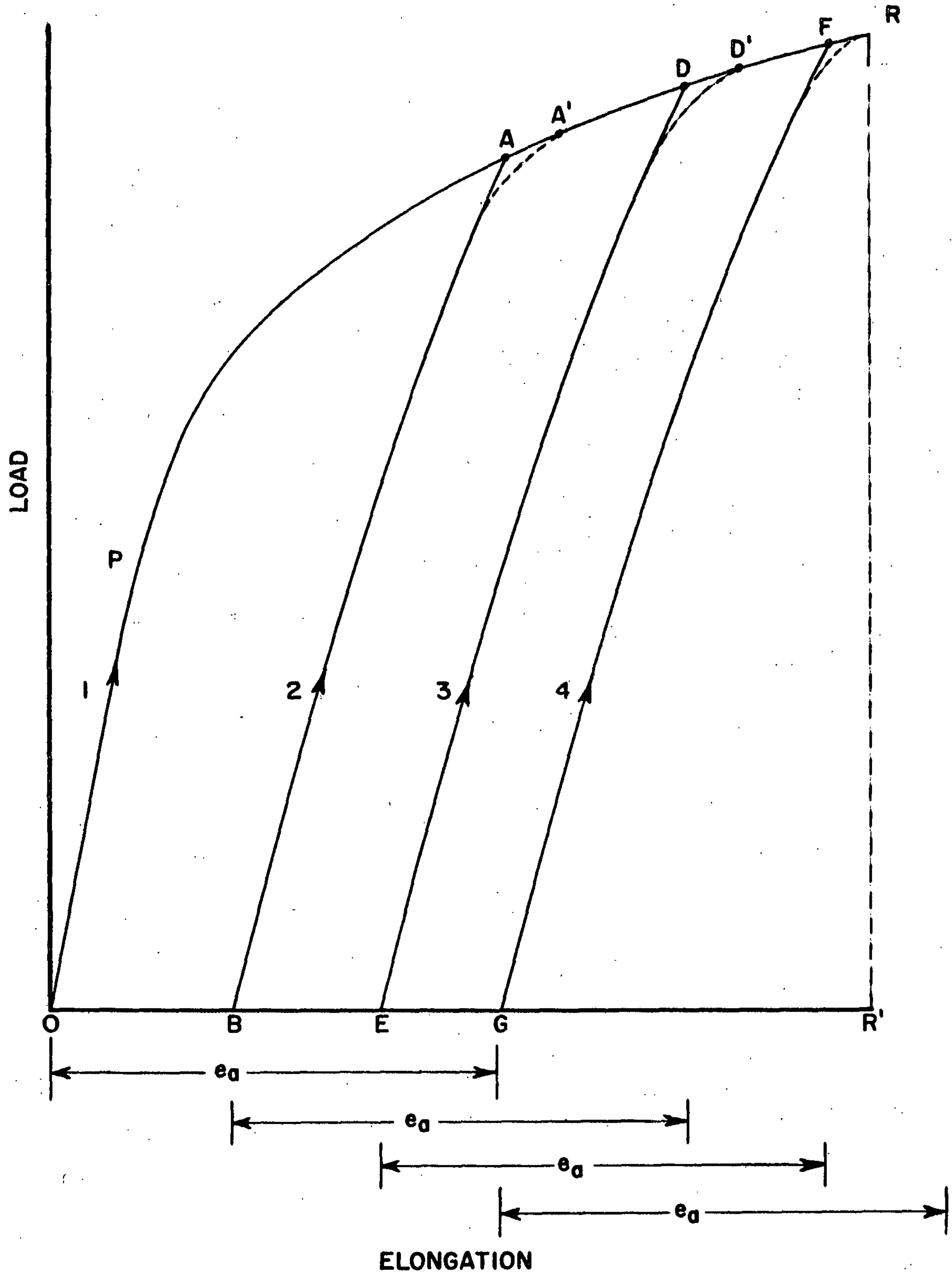


Figure 2. Idealization of Repeated Tension Load-Elongation Curve

Let it be assumed for the present that the shape of the reload curve $\overline{BA}$ of Fig. 2 and its angular orientation with respect to the axes is known along with the virgin curve $\overline{OPAR}$. For a given applied strain, e_a, the terminal point A of the first application of strain may be established. Knowing the shape and angular orientation of the reload curve $\overline{BA}$, one could graphically construct the absolute location of this reload curve by demanding that it pass through A. In so doing, the point B is established, and moreover the nonrecoverable elongation $\overline{OB}$ of the first cycle is determined. Having located point B, the terminal point D of the second application of strain, e_a, may be determined. If, furthermore, the shape and orientation of the second reload curve, $\overline{ED}$, were known, its absolute location could be established by having it pass through point D. In turn, the point E and also the second increment of nonrecoverable elongation, $\overline{BE}$, would be determined. If the shapes and orientation of the subsequent reload curves were known, this process could be continued. Eventually, a stage in the construction would be reached where the contemplated application of strain, e_a, would exceed the point R, thereby denoting rupture of the paper.

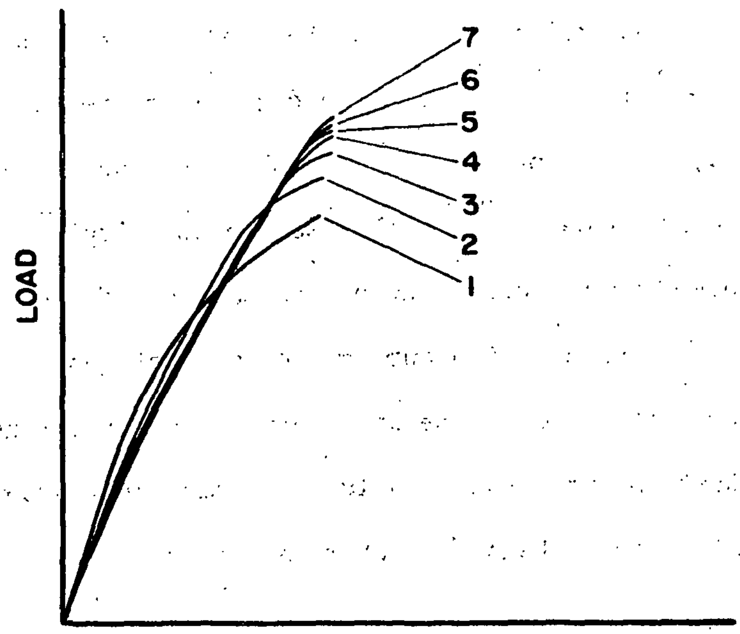
Thus, it is seen that a knowledge of the shape and angular orientation of the reload curves and the virgin curve permits construction of the repeated tension history of the paper for a specified applied strain process, and hence determines the fatigue life.

Regarding the necessary prior knowledge of the shape and orientation of the reload curves, it may be instructive to examine the nature of these curves as exhibited by repeated tension tests. For this purpose, the reload curves from repeated tension tests (constant applied elongation) of a regular 50-lb. sack paper, in- and cross-machine directions, are reproduced in Fig. 3. The reload curves for either direction are traced from a common origin to facilitate comparison of successive reload curves. The numeral with each curve is the number of the load application in the repeated tension test. It may be seen that in the case of in-machine tension (Fig. 3a) there is a more or less progressive clockwise rotation of the reload curves as the number of load applications increases. The greatest change occurs between the virgin curve and the first reload curve (i.e., second application of load). Ihrman and Andersson (3) report a similar trend for sack paper which was repeatedly strained by impacting grocery bags.

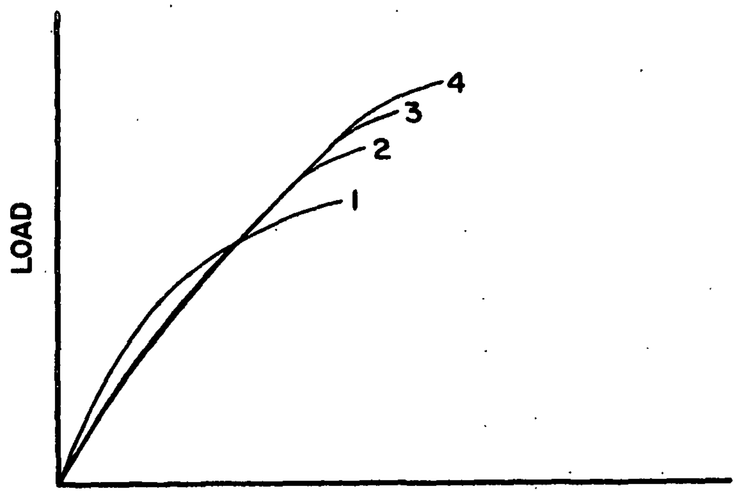
In the cross-machine direction (Fig. 3b) the reload slopes are much more nearly parallel to each other, although again there is a marked difference between the virgin slope and reload slopes. The curves of Fig. 3b agree with an observation reported in Reference (4).

Inspection of a large number of repeated tension curves for both the in- and cross-machine directions of several samples of 50-lb. regular sack paper with various levels of applied strain substantiated the trends exemplified by Fig. 3, namely, (a) the reload curves have an appreciably lesser slope than the virgin curve over comparable ranges of load, and (b) the reload curves are not always parallel among themselves.

The observation that there may not always be a unique shape of the reload curves for a given repeated tension process detracts somewhat from the utility of the proposition that the repeated tension history of a sample of paper may be predicted from a prior knowledge of the reload curves. On the



(a) In-Machine Direction



(b) Cross-Machine Direction

Figure 3. Comparison of Experimental Reload Curves for 50-lb. Regular Sack Paper (Number 1 is Virgin Curve)

other hand, a suitable approximation to all of the reload curves possibly may be afforded by, say, the average of the first and the final reload curves. The final curve probably can be found with reasonable accuracy by a simple test wherein a specimen is loaded arbitrarily near to its virgin tensile strength, unloaded and then reloaded to rupture. The approximation afforded by the mean value would introduce error of one sign during the early stages of the repeated tension history and error of the opposite sign during the later phases. It might be anticipated that the net result would be a reasonably adequate prediction of the over-all repeated tension history of the paper.

MATHEMATICAL ANALYSIS OF FATIGUE LIFE

It has been proposed that the repeated tension history, and hence fatigue life, of sack paper may be predicted by means of a graphical construction involving the virgin tensile-elongation curve, the reload curves and the magnitude of the applied strain (or work). On the expectation that a suitable approximation to the reload curves may be obtained from a limited test program for a given sample of paper, this section of the report is concerned with a mathematical description of the repeated tension process, leading to an explicit equation relating fatigue life to the macroscopic load-elongation properties of the paper.

To make the mathematical treatment tractable and the results readily practicable, it will further be assumed that the virgin load-elongation curve may be approximated satisfactorily by two straight lines and the average reload curve by one straight line, as illustrated in Fig. 4. While curvilinear approximations of the curves are indeed feasible, they would introduce considerably greater complexity into the analysis. Furthermore, for many individuals the visual comprehension of a load-elongation curve quite naturally leads to a two-straight-line equivalent.

With reference to Fig. 4, it is assumed that the following load-elongation curve parameters are known:

$\underline{S}_o$ = slope of the initial portion of the virgin curve.

$\underline{e}_o$ = elongation at the intersection of the two approximating lines (in effect, the proportional limit elongation).

$\underline{P}_o$ = load at the intersection of the two approximating lines (effective proportional limit load).

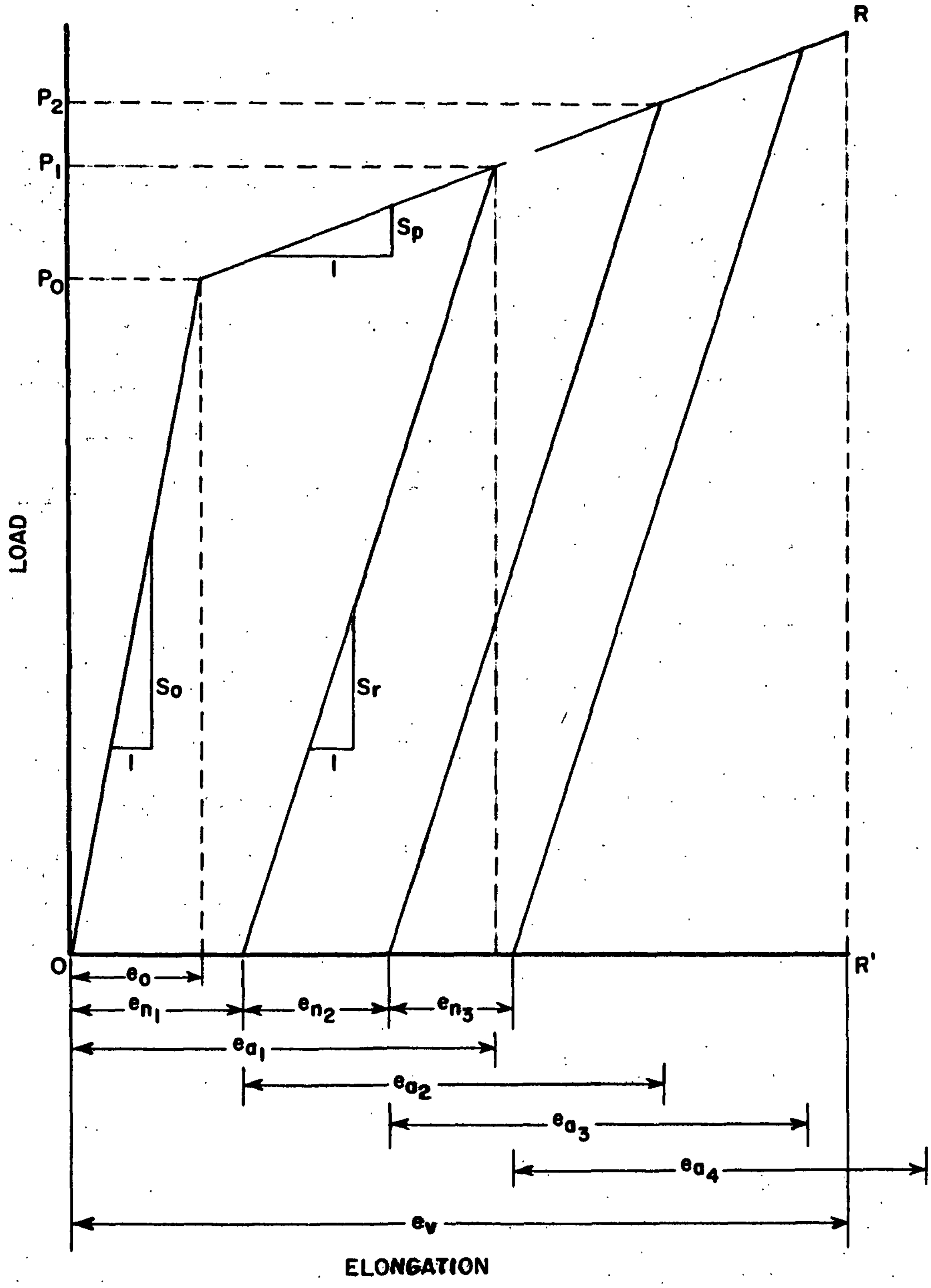


Figure 4. Approximation of Repeated Tension Load-Elongation Curve by Straight Lines

$\underline{S_p}$ = slope of plastic portion of virgin curve.

$\underline{S_r}$ = slope of average reload curve.

$\underline{e_v}$ = virgin elongation.

$\underline{e_{a_1}}$, $\underline{e_{a_2}}$, $\underline{e_{a_3}}$, $\underline{e_{a_4}}$, . . . = applied elongations (not necessarily equal).

In view of Hooke's law ($\underline{e} = \underline{PL}/\underline{Ebt}$, where $\underline{E}$ is modulus of elasticity, $\underline{b}$ is width, $\underline{t}$ is caliper, and $\underline{L}$ is length), it may be noted that $\underline{S_o}$ is the modulus of elasticity, $\underline{E}$, when the load-elongation curve is expressed in units of stress and strain. If the curve is presented in terms of load per unit width vs. unit strain, $\underline{S_o}$ is $\underline{E}$ $\underline{t}$, the tension stiffness per unit width. And if the curve is load per unit width vs. elongation, $\underline{S_o}$ is $\underline{T}$ $\underline{t}/\underline{L}$. Analogous interpretations may be given for $\underline{S_p}$ and $\underline{S_r}$ for these several ways of expressing the load-elongation behavior of a sample of paper.

The nonrecoverable elongation, $\underline{e_{n_1}}$, resulting from the first application of stress and strain is given by (see Fig. 4)

$$e_{n_1} = e_{a_1} - P_1/S_r, \tag{1}$$

where $\underline{P_1}$ is the load corresponding to the point of reversal of loading. But

$$P_1 = P_o + S_p\ (e_{a_1} - e_o) \qquad \frac{P_o}{S_r} + \frac{S_p\ (e_{a_1} - e_o)}{S_r} \tag{2}$$

provided $\underline{e_{a_1}} \ngtr \underline{e_o}$, whereupon Equation (1) becomes

$$e_{n_1} = (1 - S_p/S_r)\ e_{a_1} + (S_p/S_r)\ e_o - P_o/S_r. \tag{3}$$

However, $\underline{P_o}/\underline{S_r} = (\underline{S_o}/\underline{S_r})\ \underline{e_o}$, so that the equation for the nonrecoverable elongation after the first application of strain, $\underline{e_{a_1}}$, is

$$e_{n_1} = [1 - S_p/S_r]\ e_{a_1} + [(S_p/S_r) - (S_o/S_r)]\ e_o. \tag{4}$$

For convenience of notation, denote

$\alpha = \underline{S}_p/\underline{S}_r$ = ratio of plastic slope to reload slope

$\beta = \underline{S}_o/\underline{S}_r$ = ratio of initial slope to reload slope,

whereupon the nonrecoverable elongation is

$$e_{n_1} = (1 - \alpha)\, e_{a_1} + (\alpha - \beta)\, e_o. \tag{5}$$

The increment in nonrecoverable elongation, $\underline{e}_{n_2}$, resulting from the second application of elongation, $\underline{e}_{a_2}$, is

$$e_{n_2} = e_{a_2} - P_2/S_r\,. \tag{6}$$

But

$$\begin{aligned} P_2 &= P_o + S_p\,(e_{a_2} + e_{n_1} - e_o) \\ &= P_o + S_p\,[e_{a_2} + (1 - \alpha)\, e_{a_1} + (\alpha - \beta - 1)\, e_o], \end{aligned} \tag{7}$$

whereupon the increment of nonrecoverable elongation, $\underline{e}_{n_2}$, is

$$e_{n_2} = (1 - \alpha)\, e_{a_2} - (1 - \alpha)\, \alpha\, e_{a_1} + (1 - \alpha)(\alpha - \beta)\, e_o\,. \tag{8}$$

Similarly, it may be shown that the third increment in nonrecoverable strain is

$$\begin{aligned} e_{n_3} = {} & (1 - \alpha)\, e_{a_3} - (1 - \alpha)\, \alpha\, e_{a_2} - (1 - \alpha)^2 \alpha\, e_{a_1} \\ & + (1 - \alpha)^2 (\alpha - \beta)\, e_o\,. \end{aligned} \tag{9}$$

In general, the $\underline{k}^{th}$ increment of nonrecoverable elongation, $\underline{e}_{n_k}$, resulting from the $\underline{k}^{th}$ application of elongation, $\underline{e}_{a_k}$, is

$$\begin{aligned} e_{n_k} = {} & (1 - \alpha)\, e_{a_k} - (1 - \alpha)\, \alpha\, e_{a_{k-1}} - (1 - \alpha)^2\, \alpha\, e_{a_{k-2}} \\ & - (1 - \alpha)^3\, \alpha\, e_{a_{k-3}} - \cdots \\ & + (1 - \alpha)^{k-1} (\alpha - \beta)\, e_o \end{aligned} \tag{10}$$

where $\underline{k} = 1, 2, 3, \ldots$

With reference to Fig. 4, fatigue failure occurs on the fourth application of strain because

$$e_{n_1} + e_{n_2} + e_{n_3} + e_{a_4} > e_v \qquad (11)$$

that is, the sum of the increments of nonrecoverable elongation and the fourth applied elongation exceeds the virgin elongation. In the general case, rupture will occur on the $(\underline{k} + 1)^{st}$ application of strain if

$$e_{n_1} + e_{n_2} + \circ \circ \circ + e_{n_k} + e_{a_{k+1}} \geqq e_v \circ \qquad (12)$$

Equation (12) is the condition for fatigue failure, namely: rupture will occur when the sum of the nonrecoverable elongation increments and the final applied elongation equals or exceeds the virgin elongation. From Equations (5), (8), (9), and (10), a table may be constructed which represents the failure condition of Equation (12):

Number of Applications to Cause Rupture, $\underline{k} + 1$	Failure Condition
1	$\underline{e}_{a_1} \geqq \underline{e}_v$
2 ($\underline{k}$=1)	$\underline{e}_{n_1} + \underline{e}_{a_2} \geqq \underline{e}_v$
3 ($\underline{k}$=2)	$\underline{e}_{n_1} + \underline{e}_{n_2} + \underline{e}_{a_3} \geqq \underline{e}_v$
4 ($\underline{k}$=3)	$\underline{e}_{n_1} + \underline{e}_{n_2} + \underline{e}_{n_3} + \underline{e}_{a_4} \geqq \underline{e}_v$
○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○
i ($\underline{k}$=i-1)	$\underline{e}_{n_1} + \underline{e}_{n_2} + \circ \circ \circ + \underline{e}_{n_{i-1}} + \underline{e}_{a_i} \geqq \underline{e}_v$

Rupture occurs with that number of applications for which the inequality in the right-hand column of the table is satisfied.

It may be of interest to consider the special case where the applied elongations are all of equal magnitude ($e_{a_1} = e_{a_2} = \ldots = e_{a_k} = e_a$), inasmuch as this may correspond approximately to the constant ~~height~~ drop test of a multiwall sack. For this case, Equations (5), (8), (9), and (10) for the increments in nonrecoverable elongation become, respectively,

$$e_{n_1} = (1 - \alpha)\, e_a + (\alpha - \beta)\, e_o \tag{13}$$

$$e_{n_2} = (1 - \alpha)^2\, e_a + (1 - \alpha)(\alpha - \beta)\, e_o \tag{14}$$

$$e_{n_3} = (1 - \alpha)^3\, e_a + (1 - \alpha)^2(\alpha - \beta)\, e_o \tag{15}$$

- - - - - - - - - - - - - - - - - - - -

$$e_{n_k} = (1 - \alpha)^k\, e_a + (1 - \alpha)^{k-1}\, (\alpha - \beta)\, e_o \tag{16}$$

where $k = 1, 2, 3 \ldots$

Substituting Equations (13) through (16) into Equation (12), the condition for rupture on the $(k + 1)^{st}$ application of elongation, e_a, becomes

$$\left[\sum_{j=1}^{k} (1 - \alpha)^j\right] e_a + \left[\sum_{j=o}^{k-1} (1 - \alpha)^j\right] (\alpha - \beta)\, e_o + e_a \geqq e_v \tag{17}$$

where $k = 1, 2, 3 \ldots$

Nondimensionalizing with respect to e_v and noting that $\beta \geqq 1.0$ and $\alpha < 1.0$ so that $\alpha - \beta < 0$, the condition for fatigue rupture may be written as

$$\left[1 + \sum_{j=1}^{k} (1 - \alpha)^j\right] (e_a/e_v) - \left[\sum_{j=0}^{k-1} (1 - \alpha)^j\right] (\beta - \alpha)(e_o/e_v) \geqq 1 \tag{18}$$

where $k = 1, 2, 3 \ldots$

It may be noted from strictly algebraic considerations that

$$[1 + \sum_{j=1}^{k} (1 - \alpha)^j] \equiv \sum_{j=o}^{k} (1 - \alpha)^j \equiv \sum_{j=o}^{k-1} (1 - \alpha)^j + (1 - \alpha)^k \qquad (19)$$

whereupon Equation (18) becomes

$$[\sum_{j=o}^{k-1} (1 - \alpha)^j] [(e_a/e_v) - (\beta - \alpha)(e_o/e_v)] + (1 - \alpha)^k(e_a/e_v) \gtrsim 1 \qquad (20)$$

where $k = 1, 2, 3 \ldots$

It may be recognized that the summation in Equation (20) is geometric series (18,19), viz.

$$\sum_{j=o}^{k-1} (1 - \alpha)^j = 1 + (1 - \alpha) + (1 - \alpha)^2 + \ldots + (1 - \alpha)^{k-1} . \qquad (21)$$

The first term is unity, the common ratio is $(1 - \alpha)$ and there are k terms. The sum of the first k terms of this progression is

$$\sum_{j=o}^{k-1} (1 - \alpha)^j = \frac{1 - (1 - \alpha)^k}{1 - (1 - \alpha)} = \frac{1 - (1 - \alpha)^k}{\alpha}, \qquad (22)$$

so that Equation (20) becomes, after some algebraic manipulation,

$$(1 - \alpha)^k \leq \frac{(e_a/e_v) - \alpha - (\beta - \alpha)(e_o/e_v)}{(1 - \alpha)(e_a/e_v) - (\beta - \alpha)(e_o/e_v)} \qquad (23)$$

where $k = 1, 2, 3 \ldots$

Taking logarithms of both sides of Equation (23), the following expression is obtained for the condition that fatigue failure occurs on the $(k + 1)^{st}$ application of elongation, e_a:

$$k \gtrsim \frac{\log \left\{ \frac{(e_a/e_v) - \alpha - (\beta - \alpha)(e_o/e_v)}{(1 - \alpha)(e_a/e_v) - (\beta - \alpha)(e_o/e_v)} \right\}}{\log (1 - \alpha)} \qquad (24)$$

In Equation (24), the right-hand side of the equation is a function of the virgin and reload tensile characteristics of the sack paper and of the magnitude of the applied elongation. Thus, the right-hand side may be evaluated for a particular sample of sack paper for which these parameters are known. Equation (24) determines an integer, $\underline{k}$. The next higher integer, ($\underline{k}$+1), is the number of applications of elongation, $\underline{e}_a$, which will rupture the paper.

Inasmuch as fatigue life, $\underline{N}$, is defined to be "the number of cycles that a specimen sustains before failure under a given test condition" (6), $\underline{N} = \underline{k}$ and the expression for fatigue life is

$$N \geqslant \frac{\log \left\{ \dfrac{(e_a/e_v) - \alpha - (\beta - \alpha)(e_o/e_v)}{(1 - \alpha)(e_a/e_v) - (\beta - \alpha)(e_o/e_v)} \right\}}{\log (1 - \alpha)} \qquad (25)$$

where, by way of review, $\alpha = \underline{S}_p/\underline{S}_r$ = ratio of plastic slope to reload slope
$\beta = \underline{S}_o/\underline{S}_r$ = ratio of initial slope to reload slope.

The significance of the inequality sign in Equation (25) is that, in general, the right-hand side of the expression will be a noninteger, whereupon the fatigue life, $\underline{N}$, is the next higher integer. For example, if the right-hand side evaluates to 8.3, the fatigue life, $\underline{N}$, is 9. The tenth application of elongation will rupture the paper. It is convenient to write Equation (25) without the inequality sign--it being understood that the fatigue life is the next higher integer with respect to the number calculated from the equation.

Thus, Equation (25) permits an estimate to be made of the fatigue life of a sample of sack paper under uniaxial repeated tension, providing the following parameters are known (relative to a two-straight-line approximation to the virgin tensile load-elongation curve):

$\underline{e}_o$ = elongation at proportional limit.

$\underline{S}_o$ = slope of the initial portion of the virgin load-elongation curve.

$\underline{S}_p$ = slope of the plastic portion of the virgin load-elongation curve.

$\underline{S}_r$ = slope of the (average) reload curve.

$\underline{e}_v$ = virgin elongation.

and $\underline{e}_a$ = applied elongation (constant).

It is pertinent to note that the predicted fatigue life depends upon ratios of these parameters, viz.: applied elongation, $\underline{e}_a$/virgin elongation $\underline{e}_v$; proportional limit elongation, $\underline{e}_o$/virgin elongation, $\underline{e}_v$; initial slope, $\underline{S}_o$/reload slope, $\underline{S}_r$; and plastic slope, $\underline{S}_p$/reload slope, $\underline{S}_r$. Thus, two samples of sack paper having perhaps widely differing mechanical properties (e.g., virgin elongation $\underline{e}_v$, etc.) would have the same predicted fatigue life if the values of the above mentioned ratios are the same for the two samples. Moreover, inasmuch as ratios of elongations and slopes are involved, the fatigue life may also be analyzed in terms of unit strains and/or moduli or stiffnesses.

CHARACTERISTICS OF THE FATIGUE LIFE EQUATION

VIRGIN TENSILE TEST

It may be seen from Equation (25) that when the applied elongation, e_a, is just equal to the virgin elongation, e_v, (i.e., $e_a/e_v = 1.0$), the fatigue life, N, is zero. That is, the specimen ruptures on the first application of elongation and sustains no cycles without failure. This situation is simply the virgin tensile test. Thus, the fatigue life analysis embraces the virgin tensile test.

ENDURANCE LIMIT

In the classical analysis of fatigue, the endurance limit (also termed fatigue limit) is defined as "the limiting value of the stress below which a material can presumably endure an infinite number of stress cycles . . ." (5). In the present case of repeated tension of sack paper, it is pertinent to consider the level of applied strain for which the fatigue life becomes indefinitely large for a given sample of paper.

From the simplified concepts of repeated tension, it is evident that if the applied strain, e_a, is less than the proportional limit strain, e_o, the fatigue life becomes indefinitely large. In this case (neglecting creep) the first unloading results in zero nonrecoverable elongation and the second loading repeats the same path as the first, namely, along the initial portion of the virgin curve. Similarly, each successive cycle is a repetition of the previous cycle; the loading and unloading paths merely move up and down along the initial portion of the virgin curve.

The behavior described above is probably not exact for sack paper in view of the fact that few paper samples exhibit a true linear initial portion of the load-elongation curve. Rance has shown (20) that paper displays some nonrecoverable elongation even when cycled to arbitrarily small stress or strain levels, so that successive cycles could not follow exactly the path of the virgin curve, as described above for the straight-line model of the load-elongation curve. However, the effect noted by Rance would still lead to a very high fatigue life when the applied strain is less than the proportional limit strain because the increments in nonrecoverable elongation would be modestly small.

Examination of the fatigue life equation [Equation (25)] reveals that there is yet another level of applied strain for which the fatigue becomes infinite, as discussed below. Moreover, this level of applied strain is greater than the proportional limit strain and therefore takes precedence over the latter as the endurance limit of the paper. This case will be studied in detail in the following paragraphs.

It may be seen from Equation (25) that when

$$(e_a/e_v) = \alpha + (\beta - \alpha)\ (e_o/e_v) \qquad (26)$$

the numerator of Equation (25) becomes log 0, which is $-\infty$. Inasmuch as α is less than unity, the denominator of Equation (25), $\log (1 - \alpha)$, is inherently negative. Thus, the fatigue life, $\underline{N}$, becomes $+\infty$ for the value of applied strain, $\underline{e_a}$, given by Equation (26). The endurance limit, which will be denoted by $\underline{e_\infty}$, therefore, is

$$e_\infty = \alpha\ e_v + (\beta - \alpha)\ e_o. \qquad (27)$$

That is, for a given sample of paper, there is a critical level of elongation, e_∞, which depends on the virgin elongation, proportional limit elongation and the slopes of the virgin and reload curves. If a constant level of strain less than or equal to this critical value is applied repetitively, the paper theoretically will sustain an infinite number of cycles without rupture.

That the endurance limit strain, e_∞, is indeed greater than the proportional limit strain, e_o, may be seen from the following considerations. Equation (27) may be written

$$\frac{e_\infty}{e_v} = \alpha + (\beta - \alpha)\ \frac{e_o}{e_v} \qquad (28)$$

Subtracting (e_o/e_v) from each side of Equation (28) and rearranging,

$$\frac{e_\infty}{e_v} - \frac{e_o}{e_v} = (1 - \frac{e_o}{e_v})\ \alpha + (\beta - 1)\ \frac{e_o}{e_v} \qquad (29)$$

For regular sack papers, $0 < e_o/e_v < 1.0$, $0 < \alpha < 1.0$ and $\beta \geq 1$. Therefore the right-hand side of Equation (29) is always greater than zero, i.e.,

$$\frac{e_\infty}{e_v} - \frac{e_o}{e_v} > 0 \qquad (30)$$

whereupon

$$e_\infty > e_o\ . \qquad (31)$$

Thus, the endurance limit is always greater than the proportional limit elongation.

For a geometrical interpretation of the endurance limit elongation, Equation (27) may be written in the following form (recalling that $\alpha = S_p/S_r$ and $\beta = S_o/S_r$:

$$S_r e_\infty = S_p (e_v - e_o) + S_o e_o \tag{32}$$

Each term in Equation (32) has the dimension of a tensile load and may be identified with a distance parallel to the load axis of the load-elongation curve as illustrated in Fig. 5. It may be seen from Fig. 5 that the left-hand side of Equation (32), namely, $S_r e_\infty$, is equal to the virgin tensile strength of the sack paper. The endurance limit elongation, e_∞, is that value of applied elongation which will just attain the rupture point R of Fig. 5 when reloading along the path whose slope is S_R. A greater value of applied elongation (applied repetitively) will cause rupture of the paper, while a lesser value is incapable of attaining the rupture load, that is, the fatigue life is infinite.

The endurance limit elongation, e_∞, may be recognized as the elastic component of the total virgin elongation, as was discussed in Reference (4). The state of the material at point C of Fig. 5 was described in Reference (4) as completely fatigued. Thus, the endurance limit of the paper is simply the elastic component of virgin stretch; all of the plastic stretch has been removed from the paper by previous applications of the elongation e_∞.

It is perhaps significant to point out that it was not necessary in this present study to postulate the existence of a completely fatigued state of the paper, as was done in Reference (4). Rather, the concept of a completely fatigued state arises naturally in the present study as a consequence of the basic premises of repeated tension behavior. Reviewing, these basic premises are essentially: (a) the rupture point in tension is

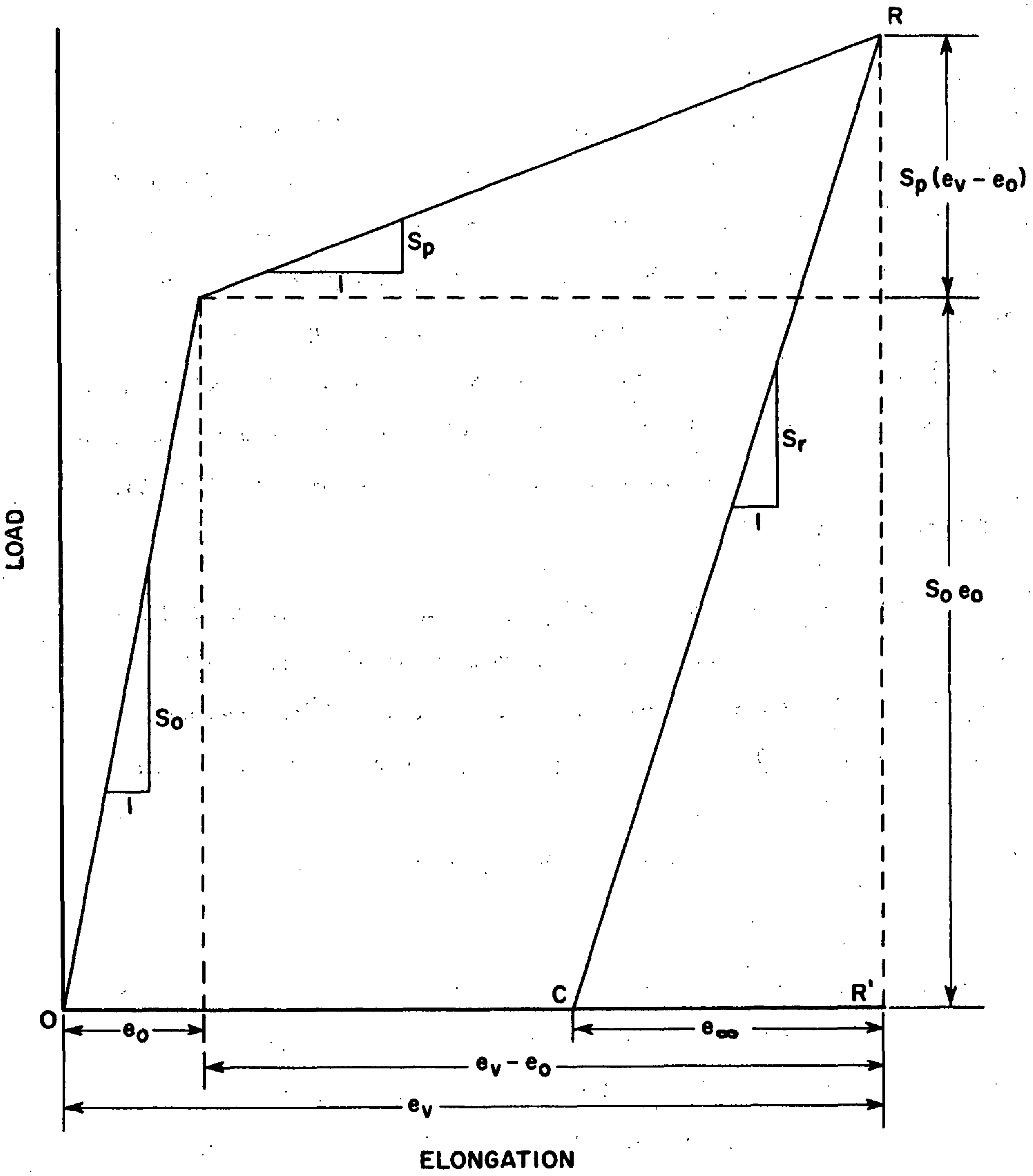


Figure 5. Interpretation of the Endurance Limit as the Elastic Component of Virgin Elongation

an invariant with respect to repeated tension processes, and (b) the reload curves in repeated tension pass through the terminal point of the previous loading and thereafter follow the virgin branch of the load-elongation curve.

A geometrical interpretation has been given for the endurance limit, e_{∞}. Later in this report an analytical treatment of the existence of the endurance limit is given. The analysis shows that when $e_a = e_{\infty}$, the increments in nonrecoverable elongation, e_{n_k}, approach zero in the limit as the number of applications approaches infinity. Inasmuch as the increments approach zero, the accumulated nonrecoverable elongation approaches a constant value (namely, $\overline{OC}$ of Fig. 5). Thus (neglecting creep) no further deterioration of the virgin stretch occurs and the paper cycles within the completely fatigued state as the number of applications become indefinitely large.

[It may be noted in passing that if the applied elongation is less than the endurance limit, e_{∞}, Equation (25) for fatigue life is no longer mathematically applicable. This limitation arises because the numerator of Equation (25) <u>before taking the logarithm</u> is negative for these values of e_a and the logarithm is not defined. Although the only restriction in the analysis of fatigue life at the outset was that $e_a \geq e_o$, is now apparent that to be mathematically meaningful, the fatigue life equation must be limited to values of e_a which are equal to or greater than the endurance limit, e_{∞}. The fatigue life of a paper when $e_a < e_{\infty}$ must be regarded as infinite on the same intuitive basis as when e_a is less than the proportional limit elongation, e_o, as discussed earlier.]

FATIGUE LIFE OF AN IDEALLY PLASTIC MATERIAL

An ideally plastic material is one whose load-elongation curve is horizontal in the plastic range (21,22), as illustrated in Fig. 6. Many regular sack papers approach this characteristic in their cross-machine tension behavior and thus the fatigue behavior of an ideally plastic material is of importance as a limiting case.

Inasmuch as $S_p = 0$ and hence $\alpha = 0$ for an ideally plastic substance, the fatigue life, N, from Equation (25) is

$$N = \frac{\log 1.0}{\log 1.0} = \frac{0}{0}, \tag{33}$$

which is indeterminate. Other means must be resorted to for evaluation of fatigue life.

Inspection of Equations (13) through (16) reveals that, in this case, all of the increments of nonrecoverable elongation, e_{n_k}, are equal and have the magnitude

$$e_{n_k} = e_a - \beta e_o . \tag{34}$$

Equation (18), which is the condition for fatigue rupture on the $(k+1)^{st}$ application of elongation, becomes

$$(1 + k)(e_a/e_v) - k \beta (e_o/e_v) \geq 1 \tag{35}$$

where $k = 1, 2, 3 \ldots$

Solving Equation (35) for k and recalling that $k = N$ = fatigue life, a determinate expression for fatigue life is obtained, viz.

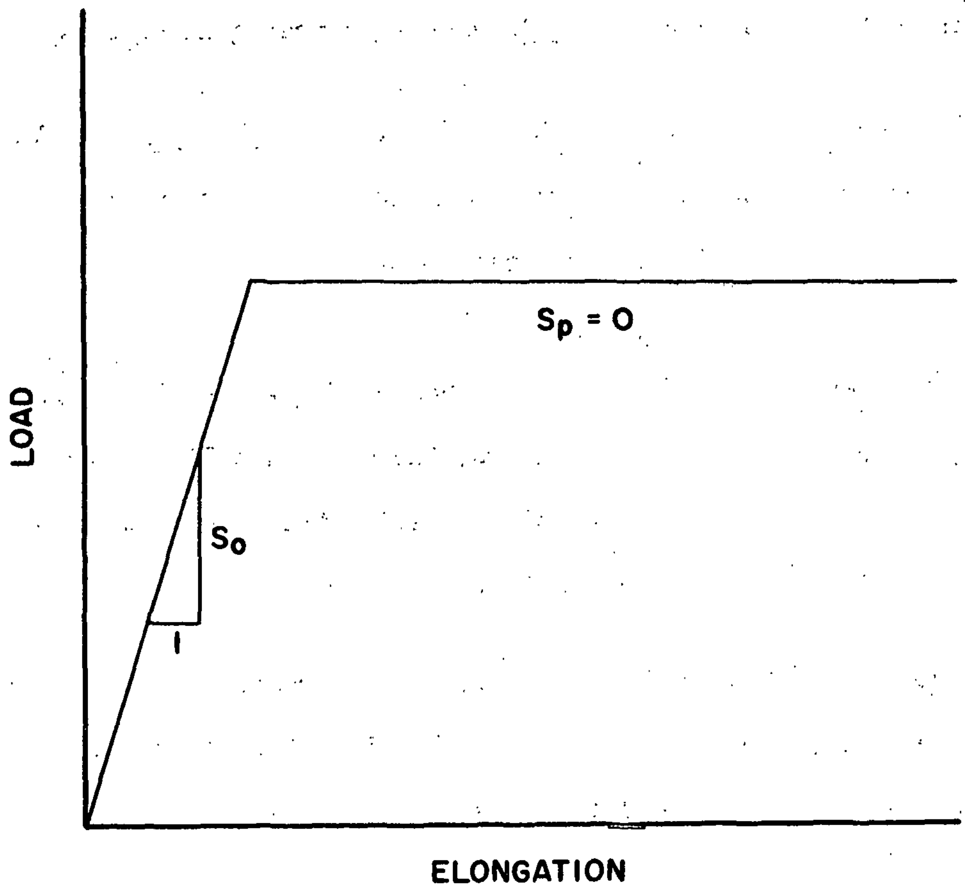


Figure 6. Load-Elongation Curve of an Ideally Plastic Material

$$N = \frac{1 - (e_a/e_v)}{(e_a/e_v) - \beta\ (e_o/e_v)}\ , \tag{36}$$

where it is to be understood that N is taken as the next higher integer if the right-hand side of Equation (36) evaluates to a non-integer.

It may be seen from Equation (36) that if $e_a = e_v$, the fatigue life is zero; this case is the virgin tensile test. The endurance limit elongation, e_∞, for an ideally plastic material is seen to be

$$e_\infty = \beta\ e_o \tag{37}$$

inasmuch as N becomes infinite for this value of e_a. Recalling that $\beta = S_o/S_r \geq 1.0$ for regular sack papers, the endurance limit is equal to or somewhat larger than the proportional limit strain, depending on the ratio of the initial and reload slopes.

All other factors being equal, the endurance limits of a viscoelastic material ($\alpha \neq 0$) and an ideally plastic material ($\alpha = 0$) are related as follows [see Equations (27) and (37)]:

$$\frac{(e_\infty)_{\text{viscoelastic}}}{(e_\infty)_{\text{ideally plastic}}} = \frac{\alpha\ e_v + (\beta - \alpha)\ e_o}{\beta\ e_o}$$

$$= 1 + \frac{\alpha}{\beta}\left(\frac{e_v}{e_o} - 1\right)$$

$$> 1.0 \tag{38}$$

(inasmuch as $e_v/e_o > 1.0$). Thus, the endurance limit of a viscoelastic material is always higher than that of an ideally plastic material if the two materials are otherwise alike with regard to virgin stretch, proportional limit strain, and initial and reload slopes. An implication of this result is that the more the cross-machine load-elongation curve departs from being "flat-topped" (all other pertinent factors being equal), the higher will be its endurance limit under repeated applications of constant elongation. A high endurance limit is, of course, desirable because the material is capable of withstanding correspondingly high levels of repeated strain without rupture.

GRAPHICAL ILLUSTRATION OF FATIGUE LIFE EQUATION

A representative graph of theoretical fatigue life as a function of applied elongation (relative to virgin elongation) is shown in Fig. 7. These curves were calculated from Equation (25). For this purpose it was assumed that $\beta = S_o/S_r = 1.25$ and $e_o/e_v = 0.25$; these values were suggested by exploratory work with regular 50-lb. sack paper. Various values of $\alpha = S_p/S_r$ were selected, representing differing degrees of plasticity.

The ordinates and abscissae of Fig. 7 are the same as in the conventional S-N diagram (5,6,23,24) of classical fatigue analysis, except that the applied elongation, e_a, (analogous to applied stress in classical fatigue) is expressed as a ratio of the virgin elongation. Moreover, the fatigue life scale is limited to 25, rather than the many millions of classical fatigue, reflecting thereby the interest of the current study in fatigue lives of the same order of magnitude as the fatigue life of a sack in the laboratory impact test.

As an example of the use of Fig. 7, suppose that the ratio α were 0.10, i.e., a nearly flat-topped load-elongation curve such as is characteristic of the cross-machine direction of sack paper. If an elongation equal to 45%

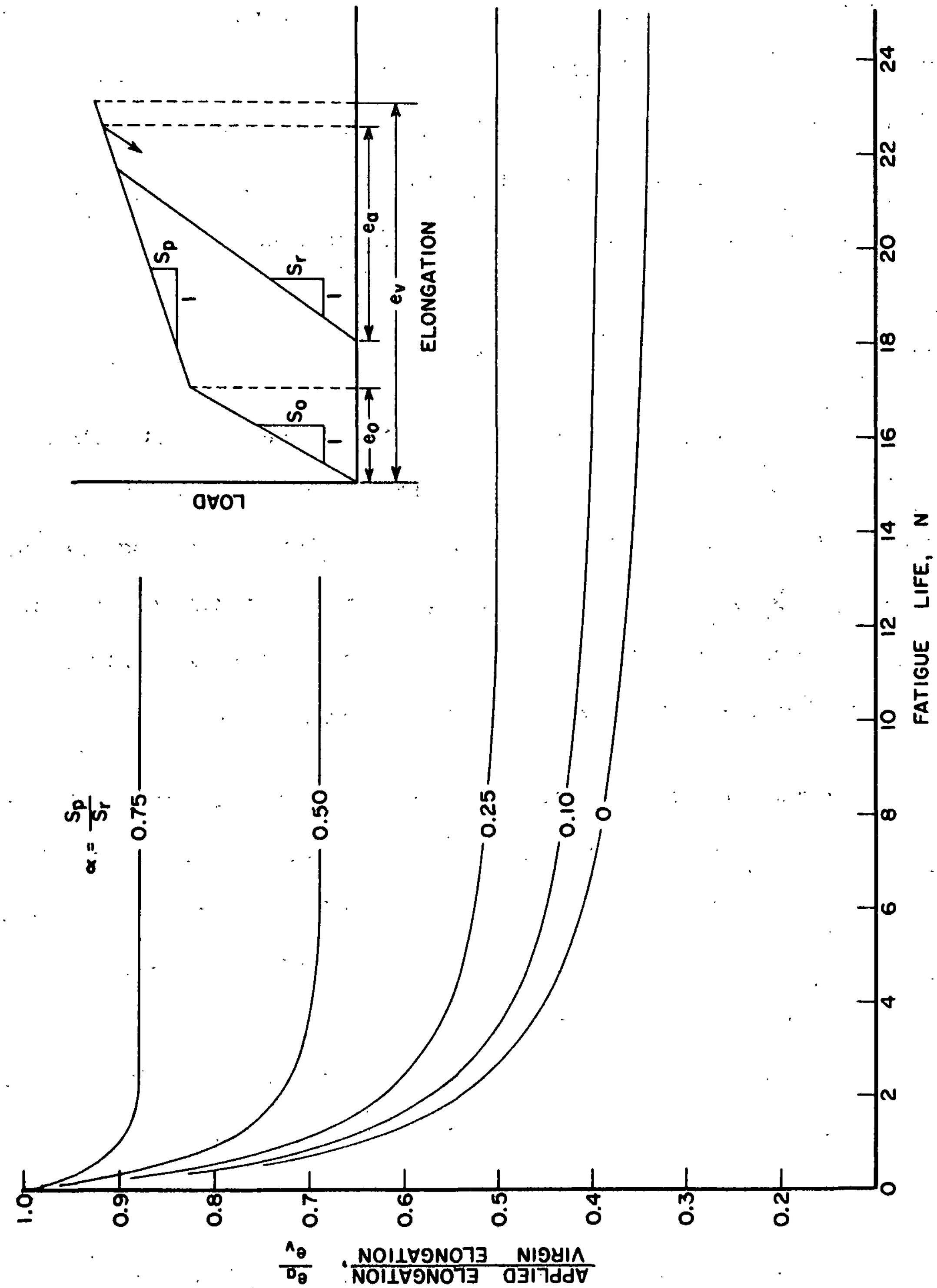


Figure 7. Theoretical Relationship Between Fatigue Life and Applied Elongation (S-N Diagram) for Various Degrees of Plasticity

($\beta = S_o/S_r = 1.25$, $e_o/e_v = 0.25$)

of the virgin elongation were repetitively applied to the paper (that is, $e_a/e_v = 0.45$), Fig. 7 indicates that the fatigue life of this hypothetical sample of paper would be six; that is, on the average, the paper could safely sustain six applications of elongation but would rupture on the seventh application. If the applied elongation were increased to 50% of the virgin elongation, the fatigue life would be reduced to four (i.e., the next higher integer above the value read from the curve).

It may be noted that any value of applied elongation greater than about 70% of the virgin elongation (for this particular case) causes the fatigue life to be reduced to unity. That is, the paper can withstand only one application of these high levels of strain; the paper will fail on the second application. In the particular instance of $e_a/e_v = 1.0$, the curve (and all of the other curves also) passes through the point $N=0$. This case is simply the virgin tensile test; the paper fails on the first application of strain--its fatigue life is zero.

At the other extreme of the $\alpha = 0.10$ curve, the predicted fatigue life becomes large very rapidly as e_a/e_o diminishes from 0.45. The fatigue life becomes theoretically infinite at the endurance limit, which in this instance is $e_\infty = 0.3875\ e_v$, that is, about 39% of the virgin elongation.

The curve labeled $\alpha = 0$ corresponds to an ideally plastic material. As discussed in a preceding section, it has the lowest endurance limit of any of those under consideration in Fig. 7, namely $e_\infty = 0.3125\ e_v$. It may be seen that the fatigue life of the ideally plastic material approaches infinity relatively more slowly than the other curves as the applied elongation approaches the endurance limit. At the other extreme, a material possessing a very low degree of plasticity, such as represented by the $\alpha = 0.75$ curve, has a very high endurance limit ($\underline{e}_{\infty} = 0.875\ \underline{e}_v$ for $\alpha = 0.75$) and the fatigue life approaches infinity very rapidly.

Available experience indicates that regular 50-lb. sack papers may be expected to have plasticity factors in the range of $\alpha = 0.10$ to $\alpha = 0.50$. In this range, fatigue lives between one and twenty-five correspond to applied elongations ranging from approximately 40 to 80% of the virgin elongation for papers represented by the curves of Fig. 7. This range of applied strains agrees favorably with experimental results of Reference (4) [c.f. Fig. 15 and 18 of (4)].

It may be remarked that, except for the magnitudes involved, the fatigue curves of Fig. 7 are reminiscent of the traditional S-N diagrams of the fatigue of metals or other materials.

REMARKS ON CHARACTERIZING FATIGUE BEHAVIOR

Figure 7 lends emphasis to the strong dependence of fatigue life on the level of the applied elongation. The relationship between these two quantities is highly nonlinear when considered over the entire range of fatigue life. In the neighborhood of the endurance limit the fatigue life can be expected to be very sensitive to small variations in the applied elongation. Obviously the fatigue life of a particular sample of paper can only be specified in terms of the particular level of applied elongation which is employed to evaluate it--either theoretically or experimentally. The question arises, therefore, as to what is the most expeditious way to characterize the fatigue of a sample of paper?

Sack paper fatigue testers in current use evaluate fatigue life at some arbitrary, and generally unknown, level of applied stress or strain. For example, both the Frag and Thwing-Albert fatigue testers apply essentially constant input energies to the paper specimen (the magnitude differing between the two types of testers). Thus, the numerical value of fatigue life is a characteristic of the particular tester as well as the paper properties. (It should be remarked that Ragossnig offers a theoretical conversion for the Frag test results, whereby the experimental fatigue life may be expressed in terms of an absolute strength of the sheet before any fatigue deterioration has occurred.) Although these two types of tests yield differing estimates of fatigue life, Fig. 7 indicates that it might be expected that the fatigue lives of a collection of samples should be ranked in approximately the same order by both types of tests. (The ranking should be exactly the same if $\beta = \underline{S}_o/\underline{S}_r$ and $\underline{e}_o/\underline{e}_v$ were identical for all samples of the collection.) It was found in Reference (25), however, that the two types of tests were not well correlated, exhibiting correlation coefficients of 0.73 for in-machine direction and 0.36 for the cross-machine direction. A complicating consideration on this point is that the Frag tester evaluates uniaxial fatigue while the Thwing-Albert evaluates biaxial fatigue.

It may be of interest to consider an alternate way of characterizing the fatigue behavior of a sample, namely, in terms of its endurance limit, as is frequently done with other materials (23). As discussed above, the endurance limit is the critical value of applied elongation below which the fatigue life is infinite. In terms of Fig. 7, the endurance limit is the value of applied elongation for which each curve becomes horizontal.

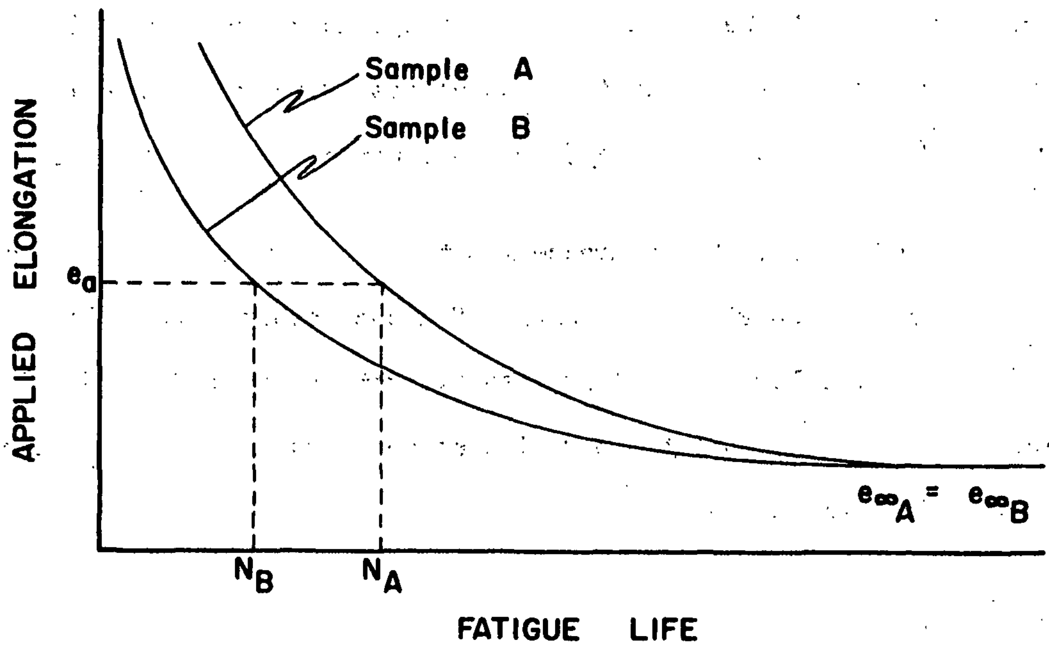
Although the endurance limit is laborious to evaluate experimentally, its theoretical make-up is quite simple and concise, namely,

$$e_{\infty} = \alpha\, e_v + (\beta - \alpha)\, e_o \tag{27}$$

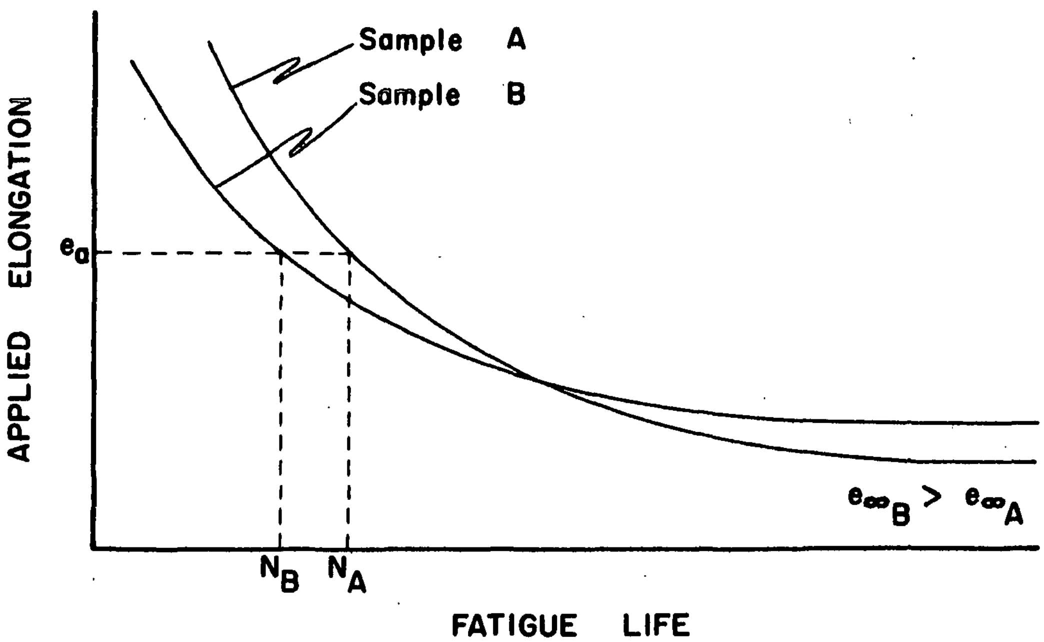
where $\alpha = S_p/S_r$ has the ready interpretation of a plasticity factor. It may be noted that the endurance limit, e_{∞}, is linear in each of the load-elongation parameters taken individually (that is α, β, e_o and e_v). Thus, the effect of any one of these variables individually on the endurance limit may be easily assessed. This proportionality is not to be expected for fatigue life, N, as may be seen from Equation (25).

In general, the higher the endurance limit of a sample, the better is its fatigue performance, inasmuch as applied elongations less than the endurance limit are theoretically incapable of rupturing the paper. Unfortunately, it cannot be expected that the fatigue lives of a collection of samples at an arbitrary applied elongation (greater than the endurance limit) will be ranked in exactly the same order as their endurance limits. This point may be illustrated by the two hypothetical samples of Fig. 8a. Samples A and B have the same endurance limit, but because of differences in their load-elongation parameters (α, β, e_o and e_v) the fatigue life of Sample A approaches ∞ somewhat more slowly than Sample B. For an arbitrary elongation, e_a, applied to both samples, Sample A has the higher fatigue life, N. Thus, two samples may have the same endurance limit but differing finite fatigue lives at other values of applied elongation.

Moreover, it is conceivable that the finite fatigue lives of two samples may be inverted relative to their endurance limits, as illustrated in Fig. 8b, because of differing load-elongation parameters. Thus, there



(a) Same Endurance Limit, Differing Finite Fatigue Lives



(b) Inversion of Finite Fatigue Lives and Endurance Limits

Figure 8. Hypothetical Examples of the Relationship Between Endurance Limit and Finite Fatigue Life

cannot be expected to be a one-to-one correspondence between fatigue life and endurance limit, except in the special case where the load-elongation parameters (α, β, $\underline{e}_o/\underline{e}_v$ and $\underline{e}_a/\underline{e}_v$) are constant or very nearly so.

Nonetheless, the endurance limit may have utility for characterizing the fatigue behavior of a sample of a sack paper because (a) it is independent of the level of the applied elongation (or work) and (b) its theoretical dependence on the load-elongation properties of the paper is relatively uncomplicated.

DETERIORATION OF VIRGIN STRETCH BY REPEATED TENSION

The reason that sack paper fails in repeated tension is that each application of load results in an increment of nonrecoverable elongation and thus deteriorates the stretch which is available in the paper. A similar statement may be made with respect to tensile energy absorption (i.e., tensile work). Tensile strength, on the other hand, does not deteriorate appreciably.

An earlier study (4) was concerned with an experimental investigation of deterioration in the tension properties of regular sack paper when it was subjected to various repeated tension processes in a laboratory uniaxial tensile tester. It may be recalled that the residual stretch (and work and tensile strength) was evaluated as a function of the number of loadings to which the specimen had been subjected. The results were presented as graphs of residual stretch vs. number of loadings for various arbitrary levels of applied strain or stress [see Fig. 14 through 19 of Reference (4)]. A dominant characteristic of the residual stretch (and work) curves was that the residual stretch decreased rapidly with the first several applications of load but the rate of deterioration diminished with increasing number of applications--that is, the aforementioned curves were concave upward.

The deterioration characteristics of sack paper tension properties are also amenable to theoretical analysis in terms of the concepts presented in this report. Denoting the residual elongation after k applications of elongation, e_a, by the symbol e_R, the following relationship is obtained between residual elongation, virgin elongation, e_v, and the increments of nonrecoverable elongation, e_{n_k}:

$$e_R = e_v - \sum_{j=1}^{k} e_{n_j}, \quad k = 1, 2, 3 \ldots \qquad (39)$$

Substituting from Equations (13), (14), (15), and (16), the residual elongation (for the case of constant applied elongation e_a) becomes

$$e_R = e_v - [\sum_{j=1}^{k} (1-\alpha)^j]\, e_a + [\sum_{j=0}^{k-1} (1-\alpha)^j]\, [\beta-\alpha]\, e_o\,. \qquad (40)$$

After some algebraic manipulation closely paralleling that involved in the development of Equation (23) in a preceding section of this report, the residual elongation (expressed as a ratio of the virgin elongation) is given by

$$\frac{e_R}{e_v} = 1 - \left[\frac{[1-(1-\alpha)^k]}{\alpha}\right] [(1-\alpha)(e_a/e_v) - (\beta-\alpha)\,(e_o/e_v)\,] \qquad (41)$$

where $k = 1, 2, 3 \ldots$

A representative graph of Equation (41) is presented in Fig. 9. This graph is a plot of residual elongation *versus* number of applications for several levels of applied elongation. Arbitrary levels of e_o/e_v, α and β have been selected, although their magnitudes have been suggested by experimental data on regular sack paper which are to be presented in a forthcoming report. The ordinates and abscissae of Fig. 9 are the same as were employed in Fig. 14 to 19 of Reference (4), except that Fig. 9 gives the residual stretch after k applications while the graphs of the Reference (4) gave the residual stretch which was available on the $(k + 1)^{st}$ application; that is, the abscissae differ by one unit.

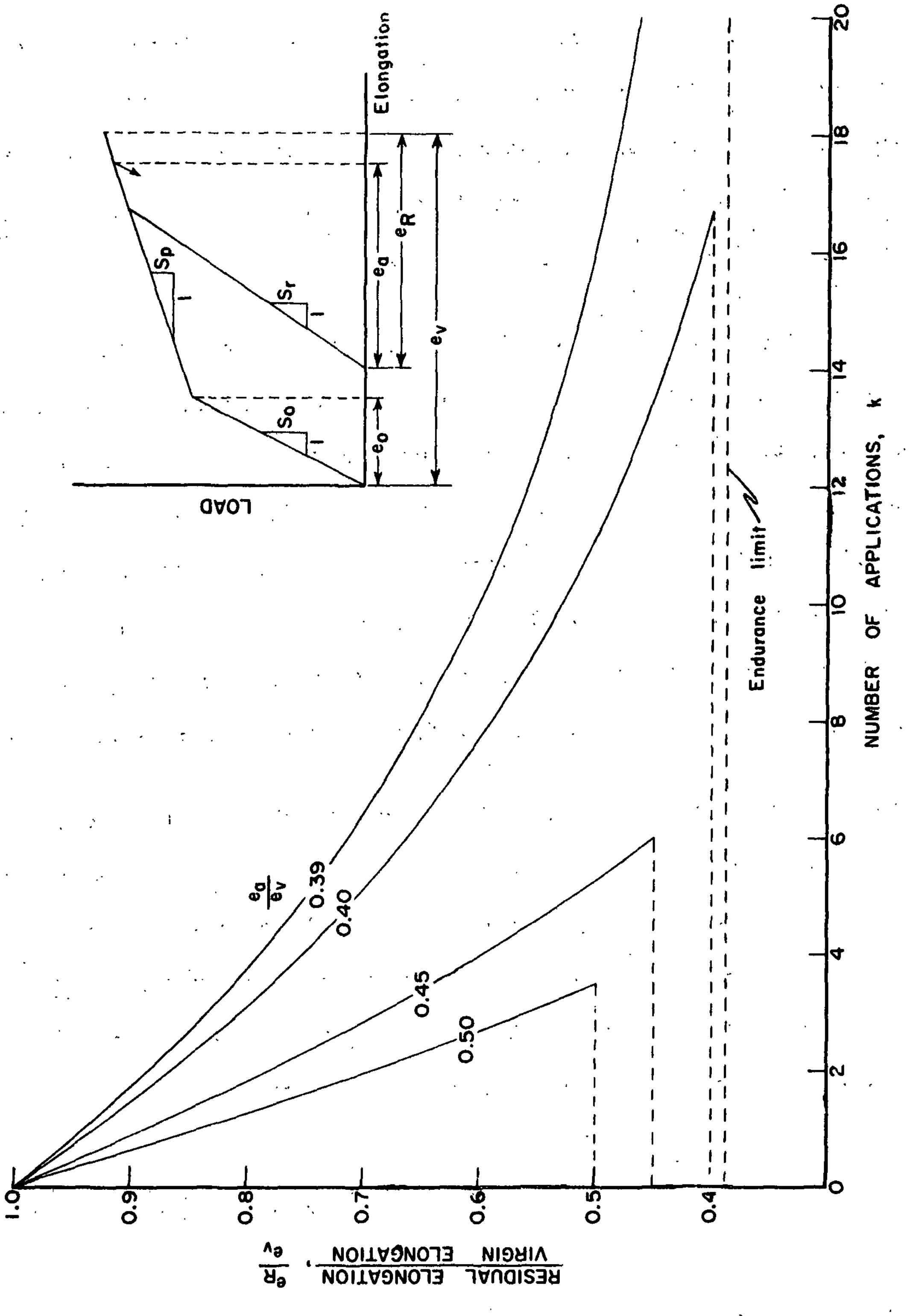


Figure 9. Theoretical Relationship Between Residual Elongation and Number of Applications of Constant Elongation

($\alpha = S_p/S_r = 0.10$, $\beta = S_o/S_r = 1.25$, $e_o/e_v = 0.25$)

Comparison of Fig. 9 with the deterioration graphs of Reference (4) reveals that the theoretical curves are of the same shape as the experimental curves. That is, both sets are concave upward. The agreement in even this limited aspect lends credence to the underlying concepts upon which the theoretical analysis of this report is based.

It may be well to point out again an observation which was made in Reference (4) relative to the shape of the deterioration curves for laboratory uniaxial tension. The concavity of these deterioration curves is of opposite sense to those of Ragossnig (26) for the theoretical strength deterioration of laterally impacted sack paper. Whether the aforementioned author's concept of strength deterioration is simply not in agreement with experimental results or whether there is a genuine difference between static and dynamic tension deterioration behavior are questions which compel further study.

It may be shown readily that the theoretical deterioration curves stemming from the concepts presented in this report will always be concave upward. In this regard it may be recalled that the concavity of a function, $y = f(x)$, is dependent on the sign of the second derivative, d^2y/dx^2: concave upward if the derivative is positive and concave downward if the derivative is negative. From Equation (41) the second derivative is found to be:

$$\frac{d^2(e_R/e_V)}{d\,k^2} = \frac{(1-\alpha)^k}{\alpha}\,[\log_e(1-\alpha)]^2\,[(1-\alpha)(e_a/e_V) - (\beta-\alpha)(e_o/e_V)]. \qquad (42)$$

Inasmuch as $\alpha < 1.0$ for regular sack papers, the factors $(1-\alpha)^k$ and $[\log_e(1-\alpha)]^2$ in Equation (42) are positive definite. Recalling that the endurance limit is given by $(e_{\infty}/e_V) = \alpha + (\beta-\alpha)(e_o/e_V)$ [see Equation (27)], the remaining factor in Equation (42) may be rewritten

$$(1-\alpha)(e_a/e_v) - (\beta-\alpha)(e_o/e_v) = [1-(e_a/e_v)]\ \alpha + (e_a/e_v) - (e_\infty/e_v)$$

$$> (e_a/e_v) - (e_\infty/e_v)$$

because $[1-(e_a/e_v)]\alpha$ is necessarily positive

$$> 0 \qquad (43)$$

because e_a is never less than e_∞ for the case of finite fatigue life.

Thus, all factors in Equation (42) are necessarily positive, which means that the deterioration curve, such as illustrated in Fig. 8, is always concave upward.

Consideration of the deterioration of the virgin elongation provides an opportunity for constructing an independent analytical proof of the existence of an endurance limit (greater than the proportional limit elongation) for the model of repeated tension employed in this study. If the applied elongation $\underline{e}_a$ is set equal to $\alpha\ \underline{e}_v + (\beta-\alpha)\ \underline{e}_o$ (which has previously been denoted $\underline{e}_\infty$), the residual elongation after $\underline{k}$ applications becomes [from Equation (41)]:

$$\frac{e_R}{e_v} = 1 - [1 - (1-\alpha)^k]\ [(1-\alpha) - (\beta-\alpha)\ (e_o/e_v)]. \qquad (44)$$

The limit of $(\underline{e}_R/\underline{e}_v)$ as the number of applications, $\underline{k}$, approaches infinity is

$$\lim_{k \to \infty} \left(\frac{e_R}{e_v}\right) = 1 - [(1-\alpha) - (\beta-\alpha)(e_o/e_v)] \qquad (45)$$

inasmuch as the $\lim_{k\to\infty} (1-\alpha)^k = 0$ (27).

Equation (45) may also be written

$$\lim_{k \to \infty} \left(\frac{e_R}{e_v} \right) = \alpha + (\beta - \alpha) \, (e_0/e_v)$$

$$\doteq \frac{e_\infty}{e_v} \tag{46}$$

in view of Equation (27). Thus

$$\lim_{k \to \infty} e_R = e_\infty, \tag{47}$$

when the applied elongation is set equal to e_∞, meaning that for the selected value of e_a (namely e_∞) the residual elongation approaches e_∞ in the limit as the number of applications becomes infinitely large. Thus, e_∞ is the endurance limit, as deduced in an earlier section of the report. In terms of Fig. 8, the deterioration curve becomes horizontal at $(e_a/e_v) = (e_\infty/e_v)$ for infinitely large values of k (i.e., a horizontal asymptote). If the applied elongation, e_a, is greater than the endurance limit, the residual elongation curve will take on a value equal to the applied elongation at a finite value of k (as in the case of three of the curves of Fig. 9) and rupture will occur on the succeeding application of load, as discussed in Reference (4).

PROPOSALS FOR FUTURE WORK

A forthcoming progress report will deal with the experimental verification of this theory for the behavior of sack paper in repeated uniaxial tension. This work is now in progress. It may be noted that there are two differing experimental approaches to validating the theory. One is to verify the fatigue life equation [namely, Equation (25)] and the other is to verify the residual elongation equation [namely, Equation (41)]. Fatigue life experimentation requires testing specimens in repeated tension until they fail, whereas residual elongation studies demand subjecting specimens from a given sample to varying numbers of applications and thereafter determining their residual elongations. The latter type of testing formed the basis for Reference (4).

It may be recalled that the derivation of the fatigue life equation was started with the general case of arbitrary, nonconstant applied elongations and then specialized to the case of constant applied elongations. As noted earlier, this specialization possibly may be meaningfully related to the constant-height sack drop test. Moreover, the special case is straightforward in terms of the mathematical development and permitted carrying through the analysis to a useful conclusion. It should be recognized, however, that a similar type of development should be possible for any other definable pattern of applied elongations. For example, if the repeated tension process were to be specified as progressively increasing applied elongation, whereby each applied elongation bears some stipulated relationship to the preceding applied elongation, it should be possible to carry out the analysis to the point of obtaining the fatigue life equation (and the residual elongation equation if that is of interest).

The opinion is held in this laboratory that mechanics of sack impact probably can be described most aptly in terms of energy processes, in so far as the behavior of the sack paper is concerned. The choice of constant applied elongation as the repeated tension process in this present study (rather than applied energy) was motivated by two considerations. First, the experimental results of Reference (4) indicated that the constant energy and constant elongation processes are equivalent for regular sack paper. That is, applying a constant level of energy repeatedly to sack paper was equivalent to applying a constant level of elongation repeatedly, and vice versa. Thus, the results of this analysis may be assumed to apply equally well to these two types of repeated tension processes in the case of regular sack paper. Secondly, it is believed that it is easier, and thereby more instructive, to visualize graphically the concepts of repeated tension in terms of elongation rather than energy, because the former involves distances along one axis of the load-elongation curve while the latter involves areas of trapezoids and polygons.

There is no reason, however, why a similar theoretical analysis can not be carried out for an arbitrary repeated tension process defined in terms of applied energy. It may be anticipated that the algebraic development of the theory would be more laborious for an applied energy process than is the present analysis for repeated elongation, but the final results may be no more complicated.

In this latter respect, it is appreciated that the fatigue life equation is perhaps not as simple as may be desired for practical application to the problems of sack paper manufacture and end-use (assuming that the theory is valid). But it should be recognized that the theory deals with a very complex phenomenon. It is perhaps noteworthy that even at this stage in the development of a repeated tension theory, there is the prospect that the phenomenon can be described in terms of five load-elongation properties and one parameter representative of the loading process.

Needless to say, if the theory proves to be valid, every reasonable attempt will be made to further simplify it. Even at the present early stage of the experimental confirmation there is evidence to support the hope that certain of the material property ratios entering into the fatigue life equation may be capable of being approximated by constants for various classes of sack papers. In this regard it might be noted that if the fatigue life equation can be simplified to the point of being a function of only two (rather than four) material property ratios, then the equation is capable of being presented succinctly in terms of a single graph. Such simplifications can proceed only from experimental data, of course, and it would be premature to speculate further at this time.

It should be recognized that the accuracy of the present theory is dependent on (a) the adequacy of the underlying concept of the mechanics of repeated tension, and (b) the precision of the straight-line approximation to the load-elongation curve. If the basic concepts should prove to be valid but the load-elongation curve approximation too crude, there are available more refined methods of approximating the curve. One of these is a curvilinear approximation developed by Ramberg and Osgood for metals up to their yield point (28). It has been found (29) that this method can be adapted to regular sack paper load-elongation curves throughout their entirety.

Provided the fatigue life theory presented in this report proves to be valid on the basis of experimental work, the following two broad courses of future work are recommended (in addition to those discussed in the preceding paragraph):

(a) A comprehensive study of the relationship between the fatigue life of sacks (i.e., laboratory sack impact) and the fatigue life of the sack paper in uniaxial tension. A sufficiently precise relationship of this type, in conjunction with a theory for uniaxial fatigue life of paper such as is presented in this report, would permit relating sack impact performance directly to uniaxial tension load-elongation properties of the sack paper. Knowledge of this kind should be of immediate benefit to the sack paper manufacturer and the converter.

(b) Relationship between sack performance and biaxial fatigue life of sack paper. On intuitive grounds, biaxial stress-strain properties of sack paper are expected to be more intimately related to sack performance than are uniaxial properties. It would appear desirable, therefore, to perform the necessary experimental and theoretical work for the purpose of relating sack impact performance to biaxial fatigue life of sack paper. If a significant improvement in predictive ability were forthcoming from this approach, the ensuing research efforts could be devoted to one or both of the following alternatives: (a) development of an appropriate biaxial fatigue test for use by the sack and sack paper industry, and (b) development of the relationship between biaxial and uniaxial fatigue life of sack paper, so that the test equipment and theory of uniaxial tension could be utilized for evaluating the sack performance potential of sack paper.

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